Lebanese American University **Spring 2012**

Byblos

Calculus IV
Final Exam

Date: Jun/08/2012
Duration: 2h

Name: N. El Khatib

ID:

I. **(15 Points)** Consider the vector field $\mathcal{F} = (-\sin(x+y)+2xe^{y+z})\,\mathbf{i}+(-\sin(x+y)+x^2e^{y+z})\,\mathbf{j}+(x^2e^{y+z})\,\mathbf{k}$.

a. Prove that $\mathcal{F}$ is conservative.

b. Find a potential function f, for the field $\mathcal{F}$.

c. Find the flow of $\mathcal{F}$ over the curve $\mathbf{r}(t) = \sin(t)\,\mathbf{i} + t\,\mathbf{j} + \sin(t)\,\mathbf{k}$ from $t = \pi$ to $t = 2\pi$.

a) $\frac{\partial M}{\partial Y} = -\cos(X+Y) + 2Xe^{Y+Z}$, $\frac{\partial N}{\partial X} = -\cos(X+Y) + 2Xe^{Y+Z}$ $\Rightarrow \frac{\partial M}{\partial Y} = \frac{\partial N}{\partial X}$

$\frac{\partial M}{\partial Z} = 2Xe^{Y+Z}$, $\frac{\partial P}{\partial X} = 2Xe^{Y+Z}$ $\Rightarrow \frac{\partial M}{\partial Z} = \frac{\partial P}{\partial X}$

$\frac{\partial N}{\partial Z} = x^2e^{Y+Z}$, $\frac{\partial P}{\partial Y} = X^2e^{Y+Z}$ $\Rightarrow \frac{\partial N}{\partial Z} = \frac{\partial P}{\partial Y}$

$\Rightarrow$ Conservative Field.

b) $\frac{\partial f}{\partial x} = -\sin(X+Y) + 2Xe^{Y+Z} \Rightarrow f = \cos(X+Y) + X^2e^{Y+Z} + g(Y,Z)$

$\frac{\partial f}{\partial Y} = -\sin(X+Y) + x^2e^{Y+Z} + \frac{\partial g}{\partial Y} \Leftrightarrow \frac{\partial g}{\partial Y} = 0 \Rightarrow g = g(z)$.

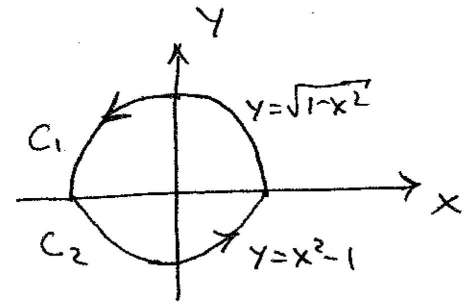
$\frac{\partial f}{\partial z} = x^2 e^{y+z} \Rightarrow g'(z) = 0 \Rightarrow$ $\boxed{f(X,Y,z) = \cos(X+Y) + X^2e^{Y+Z} + C}$

c) $A(0,\pi,0)$ $B(0,2\pi,0)$ Flow $= f(0,2\pi,0) - f(0,\pi,0) = 1 - -1 = 2$

II. **(15 Points)** Calculate the counterclockwise circulation of the vector field $\mathcal{F} = xy^3\,\mathbf{i} - y^3\,\mathbf{j}$ around the curve C which consists of the part of the parabola $y = x^2 - 1$ for $-1 \le x \le 1$ along with the positive semi-circle centered at the origin and joining $(-1, 0)$ to $(1, 0)$:

a. Using Green's theorem.

b. Directly using line integral.



$F = xy^3 i - y^3 j$

) Green's th:

$$\oint M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\frac{\partial N}{\partial x} = 0 \qquad \frac{\partial M}{\partial y} = 3xy^2$$

$$\iint_R -3xy^2\,dx\,dy = \int_{-1}^{1}\left(\int_{x^2-1}^{\sqrt{1-x^2}} -3xy^2\,dy\right)dx = \int_{-1}^{1}\left[-xy^3\right]_{x^2-1}^{\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1} \underbrace{-x(1-x^2)^{\frac{3}{2}}}_{\text{odd}} + \underbrace{x(x^2-1)^3}_{\text{odd}}\,dx = 0.$$

) Parametrization: C_1: $r_1(t) = \cos(t)\,i + \sin(t)\,j$ $\quad 0 \le t \le \pi$

C_2: $r_2(t) = t\,i + (t^2-1)\,j$ $\quad -1 \le t \le 1$

$$\oint M\,dx + N\,dy = \oint_{C_1} + \oint_{C_2}$$

$$\oint_{C_1} M\,dx + N\,dy = \int_0^{\pi} \left(\cos(t)\sin^3(t)\,(-\sin(t)) - \sin^3(t)\cos(t)\right) dt$$

$$= \left[-\frac{1}{5}\sin^5(t) - \frac{1}{4}\sin^4(t)\right]_0^{\pi} = 0.$$

$$\oint_{C_2} M\,dx + N\,dy = \int_{-1}^{1}\left(\underbrace{t(t^2-1)^3}_{\text{odd}} - \underbrace{(t^2-1)^3 2t}_{\text{odd}}\right) dt = 0.$$

$$\oint_C = \oint_{C_1} + \oint_{C_2} = 0.$$

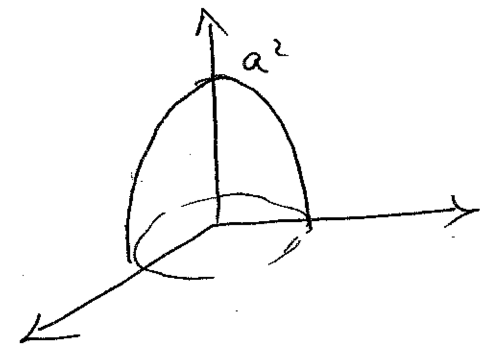
III. **(15 Points)**

a. Find the volume of the solid enclosed by the paraboloid of equation $z = a^2 - x^2 - y^2$ from above and by the plane of equation $z = 0$ from below.

b. We denote by D the region inside the paraboloid $z = 5 - x^2 - y^2$ bounded below by the plane $z = 1$ and above by the plane $z = 4$. Find the outward flux of $\mathcal{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ across the boundary of D:

   i. Using the divergence theorem.

   ii. Directly using surface integral.

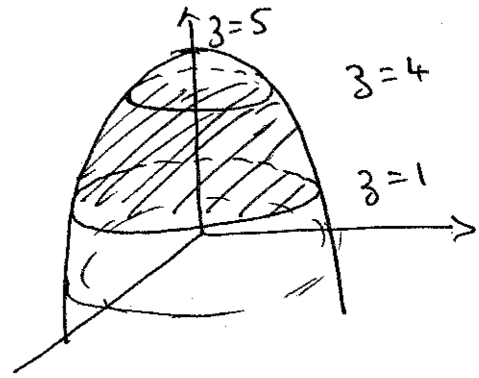


a) $V = \iiint dx\,dy\,dz$

cylindrical: $D \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le r \le a \\ 0 \le z \le a^2 - r^2 \end{cases}$

$$\iiint_D r\,dr\,d\theta\,dz = 2\pi \int_0^a \left(\int_0^{a^2-r^2} r\,dz \right) dr = 2\pi \int_0^a r(a^2 - r^2)\,dr = \left[\frac{a^2 r^2}{2} - \frac{r^4}{4} \right]_0^a 2\pi$$

$$= 2\pi \left(\frac{a^4}{2} - \frac{a^4}{4} \right) = \frac{\pi a^4}{2}.$$

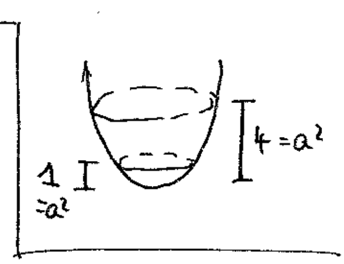


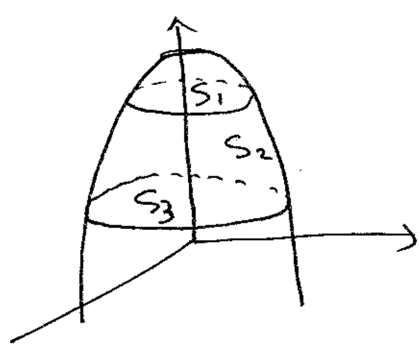
b) i) Divergence theorem:

$F = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \Rightarrow \operatorname{div} F = 3.$

$$\iiint_D 3\,dx\,dy\,dz = 3 \iiint_D dx\,dy\,dz = 3\,V_D.$$

$$= 3\left(\frac{16\pi}{2} - \frac{\pi}{2} \right) = \frac{45\pi}{2}.$$





ii) Using Surface Integral:

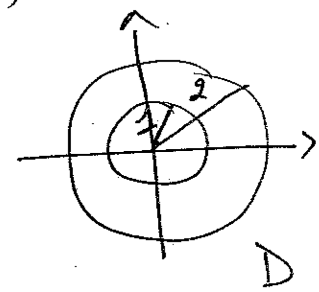
$$\iint_{S_1} F\cdot n\,d\sigma + \iint_{S_2} F\cdot n\,d\sigma + \iint_{S_3} F\cdot n\,d\sigma$$

Let's start with S_2: $x^2 + y^2 + z - 5 = 0 \Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$ outer OK ✓

$$\iint_{S_2} F\cdot n\,d\sigma = \iint (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \frac{(2x\mathbf{i} + 2y\mathbf{k} + \mathbf{k})}{|\nabla f|} \frac{|\nabla f|}{|\nabla f \cdot p|}\,dx\,dy =$$

$$= \iint_{S_2} \left(2x^2+2y^2+z\right) \frac{1}{|\nabla f \cdot k|}\, dx\,dy \qquad (\text{projection on } xy\text{-[illegible]}$$

$$= \iint_{S_2} \left(2x^2+2y^2+5-x^2-y^2\right) dx\,dy = \iint_{D} \left(x^2+y^2+5\right) dx\,dy$$



$$= \iint (r^2+5)\, r\,dr\,d\theta \qquad \begin{cases} 1 \le r \le 2 \\ 0 \le \theta \le 2\pi \end{cases}$$

$$= 2\pi \int_1^2 r^3+5r\, dr = 2\pi \left[\frac{r^4}{4}+\frac{5}{2}r^2\right]_1^2 = \frac{45\pi}{2}.$$

Now
$$\iint_{S_1} f\cdot n\,d\sigma = \iint_{S_1} (xi+yj+zk)\,\frac{k}{|k|}\,\frac{|k|}{|k\cdot k|}\,dx\,dy = \iint_{S_1} z\,dx\,dy$$

$$= \iint_{S_1} 4\,dx\,dy = 4\iint_{S_1} dx\,dy = 4\pi$$

$$\iint_{S_3} f\cdot n\,d\sigma = \iint_{S_3} (xi+yj+zk)\cdot\frac{-k}{|k|}\,\frac{|k|}{|k\cdot k|}\,dx\,dy = -\iint_{S_3} z\,dx\,dy$$

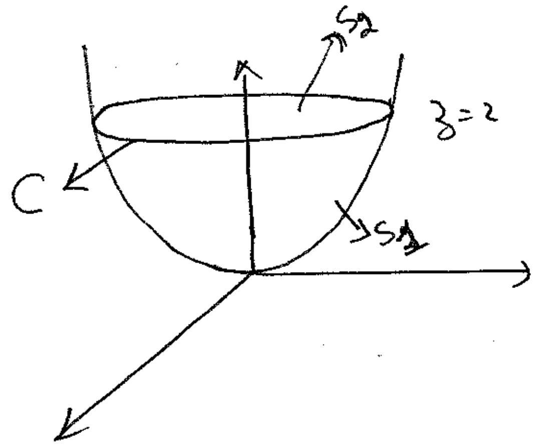
$$= -\iint_{S_3} dx\,dy = -4\pi$$

$$\iint f\cdot n\,d\sigma = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} = \frac{45\pi}{2} + 4\pi - 4\pi = \frac{45\pi}{2} \quad \text{OK}\checkmark$$

IV. **(15 Points)** Let (P) be the paraboloid of equation $x^2+y^2=2z$, and the vector field $\mathcal{F}=xy\,\mathbf{i}+xz^2\,\mathbf{j}+xy^2\,\mathbf{k}$.
Let C be the intersection of (P) with the plan of equation $z=2$.
Find the counterclockwise circulation of $\mathcal{F}$ around the curve C when viewed from above:

(a) Directly using line integral.

(b) Using Stokes' theorem in two different ways.



a) Using the line integral:

parametrization of C:

$z=2 \Rightarrow x^2+y^2=4$ $\quad C\begin{cases} x=2\cos(t) \\ y=2\sin(t) \quad 0\le t\le 2\pi \\ z=2 \end{cases}$

$$\oint F\cdot dr = \oint_0^{2\pi}\left(2\cos(t)\cdot 2\sin(t)i + 8\cos(t)j + 8\cos(t)\sin^2(t)K\right)\cdot\left(-2\sin(t)i + 2\cos(t)j + 0K\right)dt$$

$$= \int_0^{2\pi} -8\cos(t)\sin^2(t) + 16\cos^2(t)\ dt = \int_0^{2\pi} -8\sin^2(t)\cos(t) + 8 + 8\cos(2t)\ dt$$

$$= \left[-\frac{8}{3}\sin^3(t) + 8t + 4\sin(2t)\right]_0^{2\pi} = \underline{16\pi}$$

b) Fist way using Stoke's over S_1: $\quad$ Curl F $=\begin{vmatrix} i & j & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xz^2 & xy^2 \end{vmatrix}$

$\underline{\text{curl } F = (2xy-2xz)i - (y^2)j + (z^2-x)K}$

$x^2+y^2-2z=0 \quad \nabla f = 2xi+2yj-2K \quad$ NO. $\quad \nabla f = -2xi-2yj+2K$

$$\iint \text{Curl}F\cdot n\,d\sigma = \iint -4x^2y+4x^2z+2y^3+2(z^2-x)\ \frac{1}{|\nabla f|}\cdot\underbrace{\frac{|\nabla f|}{|\nabla f\cdot k|}}_{=2}\ dx\,dy$$

$$\iint -2x^2y+2x^2z+y^3+z^2-x\ dx\,dy$$

~~$\iint -2x^2y + 2x^2\frac{(x^2+y^2)}{2} + y^3 + \ldots$~~

$$\iint -2x^2y+2x^2\left(\frac{x^2+y^2}{2}\right)+y^3+\frac{(x^2+y^2)^2}{4}-x\ dx\,dy$$

Polar coords
$\begin{cases} x=r\cos(\theta) \\ y=r\sin(\theta) \end{cases}$

$$= \iint \left(-2r^3 \cancel{\cos^2\theta \sin\theta} + r^4 \cos^2\theta + 2r^3 \sin^3\theta + \frac{r^4}{4} \cancel{- r\cos\theta} \right) r\,dr\,d\theta$$

$$= \iint \left(r^5 \cos^2\theta + 2r^4 \sin^3\theta + \frac{r^5}{4} \right) dr\,d\theta$$

$$= \iint \frac{r^5}{2} + \cancel{\frac{r^5}{2}\cos 2\theta} + \cancel{2r^4 (1-\cos^2\theta)\sin\theta} + \frac{r^5}{4} \; dr\,d\theta$$

$$= \iint \frac{r^5}{2} + \frac{r^5}{4} \; dr\,d\theta = 2\pi \int_0^2 \frac{r^5}{2} + \frac{r^5}{4} \, dr = \pi \int_0^2 \frac{3r^5}{2} dr$$

$$= \pi \left. \frac{3r^6}{6.2} \right]_0^2 = \pi \left. \frac{r^6}{4} \right]_0^2 = 16\pi$$

✓

Second way: stokes' over S_2:

$$\iint (\text{curl } F . n) \, d\sigma = \iint \text{curl} F . K \frac{1}{\cancel{|K|}} . \frac{\cancel{|K|}}{\cancel{|K.K|}} dx\,dy$$

$$= \iint (z^2 - x) \, dx\,dy = \iint (4-x) \, dx\,dy = 4\iint dx\,dy - \iint x \, dx\,dy$$

$$= 16\pi - \iint r\cos\theta \, r\,dr\,d\theta = 16\pi - \underbrace{\iint \cancel{r^2\cos\theta} \, dr\,d\theta}_{=0}$$

$$= 16\pi$$

✓

V. **(15 Points)** Study the function $f(x,y) = \dfrac{xy}{(1+x^2)(1+y^2)}$ for local maxima, local minima and saddle points.

critical points:

$$\frac{\partial f}{\partial x} = \frac{y}{1+y^2}\,\frac{1+x^2-2x^2}{(1+x^2)^2} = \frac{y}{1+y^2}\,\frac{1-x^2}{(1+x^2)^2}$$

by symmetry we deduce

$$\frac{\partial f}{\partial y} = \frac{x}{1+x^2}\,\frac{1-y^2}{(1+y^2)^2}$$ hence we have the system:

$$\begin{cases} \dfrac{y(1-x^2)}{(1+y^2)(1+x^2)^2} = 0 \\ \dfrac{x(1-y^2)}{(1+x^2)(1+y^2)^2} = 0 \end{cases}$$

this gives us the set of critical points:

$(0,0)$ $(-1,1)$ $(-1,-1)$ $(1,-1)$ $(1,1)$

$$\frac{\partial^2 f}{\partial x^2} = \frac{y}{1+y^2}\cdot\frac{-2x(1+x^2)^2-(1-x^2)2(2x)(1+x^2)}{(1+x^2)^4} = \cancel{\ldots}$$

$$= -\frac{y}{1+y^2}\cdot\frac{2x(1+x^2)(3-x^2)}{(1+x^2)^4} = \frac{-y}{1+y^2}\,\frac{2x(3-x^2)}{(1+x^2)^3}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-x}{1+x^2}\,\frac{2y(3-y^2)}{(1+y^2)^3}$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{(1-x^2)(1-y^2)}{(1+x^2)^2(1-y^2)^2}$$

A $(0,0)$: $f_{xx}=f_{yy}=0$ $f_{xy}=1$

$f_{xx}f_{yy}-f_{xy}^2=-1<0$

Saddle point

B$(-1,1)$ $f_{xx}=\frac{1}{4}>0$ $f_{yy}=\frac{1}{4}$ $f_{xy}=0$.

$f_{xx}f_{yy}=\frac{1}{16}>0$

local min

$(-1,-1)$ $f_{xx}=-\frac{1}{4}<0$ $f_{yy}=-\frac{1}{4}$

$f_{xy}=0$

$f_{xx}f_{yy}-f_{xy}=\frac{1}{16}>0$ local max

D$(1,-1)$ $f_{xx}=\frac{1}{4}>0$ $f_{yy}=\frac{1}{4}$

$f_{xy}=0$ $f_{xx}f_{yy}-f_{xy}^2=\frac{1}{16}>0$

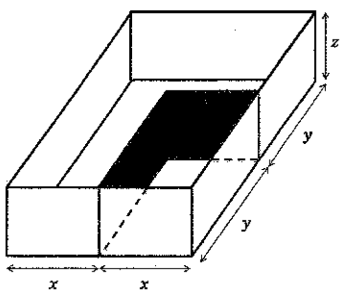
local min

E$(1,1)$ $f_{xx}=-\frac{1}{4}<0$ $f_{yy}=-\frac{1}{4}$

$f_{xy}=0$ $f_{xx}f_{yy}-f_{xy}^2=\frac{1}{16}>0$

local max.

VI. **(15 Points)** We are going to manufacture a rectangular box in order to pack an iPhone with its accessories. Apple suggests a box that has $2x$ as length, $2y$ as a width, z as a height, no top, two dividers (see the figure below) and a fixed volume of 72 cm^3. It has metal dividers, but cardboard sides. Metal costs 2 times as expensive as cardboard. For what dimensions Apple can minimize the cost of the box?



the volume $= 72$

$2x \cdot 2y \cdot z = 72 \Rightarrow \boxed{xyz - 18 = 0}$

Cost function:

Cardboard: $(4xz + 4yz + 4xy)a$

Metal: $(xz + yz)2a$

Total cost: $a(6xz + 6yz + 4xy)$

The problem is then $\begin{cases} \text{Min } f(x,y,z) = 6xz + 6yz + 4xy \\ \text{under the constraint } xyz - 18 = 0 \end{cases}$

$$\begin{cases} \nabla f = \lambda \nabla g \\ xyz - 18 = 0 \end{cases} \Rightarrow \begin{cases} 6z + 4y = \lambda yz \\ 6z + 4x = \lambda xz \\ 6x + 6y = \lambda xy \\ xyz - 18 = 0 \end{cases} \Rightarrow \begin{cases} \frac{6}{y} + \frac{4}{z} = \lambda \\ \frac{6}{x} + \frac{4}{z} = \lambda \\ \frac{6}{x} + \frac{6}{y} = \lambda \end{cases} \quad xyz = 18$$

$\frac{6}{y} - \frac{6}{x} = 0 \Rightarrow \boxed{x = y}$

$\frac{4}{z} - \frac{6}{x} = 0 \Rightarrow \boxed{z = \frac{2}{3}x}$

$\Rightarrow x \cdot x \cdot \frac{2}{3}x = 18 \Rightarrow \boxed{x = 3}$

$\boxed{y = 3}$

$\boxed{z = 2}$

VII. **(15 Points)** Find the triple integral $\iiint_D \frac{dx\,dy\,dz}{\sqrt{x^2+y^2+z^2}}$, where D is the domain limited by the two spheres

$$x^2+y^2+z^2=1 \text{ and } x^2+y^2+z^2=4.$$

$$\iiint_D \frac{dxdydz}{\sqrt{x^2+y^2+z^2}} = \iiint_D \frac{1}{\rho}\cdot \rho^2 \sin\varphi \, d\theta d\rho d\varphi$$

with D: $\begin{cases} 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi \\ 1 \le \rho \le 2 \end{cases}$

$$= \iiint \rho \sin\varphi \, d\theta d\rho d\varphi = \int_0^{2\pi} d\theta \int_1^2 \rho d\rho \int_0^{\pi} \sin\varphi \, d\varphi$$

$$= 2\pi \left[\frac{1}{2}\rho^2\right]_1^2 \cdot \left[-\cos\varphi\right]_0^{\pi} = 2\pi \cdot \left(2-\frac{1}{2}\right)\left(-\cos(\pi)+\cos(0)\right)$$

$$= 2\pi \cdot \frac{3}{2} \cdot 2$$

$$= 6\pi$$